

# Existence of least energy nodal solution with two nodal domains for a generalized Kirchhoff problem in an Orlicz Sobolev space

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## Abstract

We show the existence of a nodal solution with two nodal domains for a generalized Kirchhoff equation of the type

$$-M \left( \int_{\Omega} \Phi(|\nabla u|) dx \right) \Delta_{\Phi} u = f(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where  $\Omega$  is a bounded domain in  $\mathbf{R}^N$ ,  $M$  is a general  $C^1$  class function,  $f$  is a superlinear  $C^1$  class function with subcritical growth,  $\Phi$  is defined for  $t \in \mathbf{R}$  by setting  $\Phi(t) = \int_0^{|t|} \phi(s) ds$ ,  $\Delta_{\Phi}$  is the operator  $\Delta_{\Phi} u := \operatorname{div}(\phi(|\nabla u|) \nabla u)$ . The proof is based on a minimization argument and a quantitative deformation lemma.

## 1 Introduction

In this paper we study the existence of a nodal solution with two nodal domains for the following problem

$$(P) \quad \begin{cases} -M \left( \int_{\Omega} \Phi(|\nabla u|) dx \right) \Delta_{\Phi} u = f(u) & \text{in } \Omega, \\ u^+ \neq 0 \quad \text{and} \quad u^- \neq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

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where  $\Omega \subset \mathbf{R}^N$  is a smooth bounded domain and

$$u^+(x) = \max\{u(x), 0\} \quad \text{and} \quad u^-(x) = \min\{u(x), 0\}, \quad \text{for all } x \in \Omega.$$

Notice that, in this case,  $u = u^+ + u^-$  and  $|u| = u^+ - u^-$ . We looking for solution with exactly two nodal domains. Problem (P) with  $\phi(t) = 2$ , that is,

$$(*) \quad \begin{cases} -M \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = f(u) \text{ in } \Omega, \\ u \in H_0^1(\Omega) \end{cases}$$

is called Kirchhoff type because of the presence of the term  $M \left( \int_{\Omega} |\nabla u|^2 dx \right)$ . Indeed, this operator appears in the Kirchhoff equation, which arises in nonlinear vibrations, namely

$$\begin{cases} u_{tt} - M \left( \int_{\Omega} |\nabla u|^2 dx \right) \Delta u = g(x, u) \text{ in } \Omega \times (0, T) \\ u = 0 \text{ on } \partial\Omega \times (0, T) \\ u(x, 0) = u_0(x) \quad , \quad u_t(x, 0) = u_1(x). \end{cases}$$

The reader may consult [2], [3], [24], [39] and the references therein, for more physical motivation on Kirchhoff problem.

Before stating our main results, we need the following hypotheses on the function  $M$ .

The function  $M : \mathbf{R}_+ \rightarrow \mathbf{R}_+$  is  $C^1$  class and satisfies the following conditions:

(M<sub>1</sub>) The function  $M$  is increasing and  $0 < M(0) =: \sigma_0$ .

(M<sub>2</sub>) The function  $t \mapsto \frac{M(t)}{t}$  is decreasing.

A typical example of function verifying the assumptions (M<sub>1</sub>) – (M<sub>2</sub>) is given by

$$M(t) = m_0 + bt, \quad \text{where } m_0 > 0 \quad \text{and} \quad b > 0.$$

This is the example that was considered in [33], [40], [41], [44] and [50]. More generally, each function of the form

$$M(t) = m_0 + bt + \sum_{i=1}^k b_i t^{\gamma_i}$$

with  $b_i \geq 0$  and  $\gamma_i \in (0, 1)$  for all  $i \in \{1, 2, \dots, k\}$  verifies the hypotheses (M<sub>1</sub>) – (M<sub>2</sub>). An another example is  $M(t) = m_0 + \ln(1 + t)$ .

The hypotheses on the function  $\phi : (0, +\infty) \rightarrow (0, +\infty)$  of  $C^1$  class are the following:

( $\phi_1$ ) For all  $t > 0$ ,

$$\phi(t) > 0 \quad \text{and} \quad (\phi(t)t)' > 0.$$

( $\phi_2$ ) There exist  $l \in (\frac{N}{2}, N)$ ,  $l < m < \min\{\frac{l^*}{2}, N\}$ , where  $l^* = \frac{lN}{(N-l)}$  such that

$$l \leq \frac{\phi(t)t^2}{\Phi(t)} \leq m,$$

for  $t > 0$ , where  $\Phi(t) = \int_0^{|t|} \phi(s)sd s$ .

( $\phi_3$ ) For all  $t > 0$ ,

$$l - 1 \leq \frac{(\phi(t)t)'t}{\phi(t)t} \leq m - 1.$$

We assume that the function  $f$  is  $C^1$  class and satisfies

( $f_1$ )

$$\lim_{|t| \rightarrow 0^+} \frac{f(t)}{\phi(t)t} = 0$$

and

( $f_2$ )

$$\lim_{|t| \rightarrow \infty} \frac{f(t)}{\phi_*(t)t} = 0.$$

( $f_3$ ) There is  $\theta \in (2m, l^*)$  such that

$$0 < \theta F(t) \leq f(t)t, \quad \forall |t| > 0, \quad \text{where } F(t) = \int_0^t f(s)ds.$$

( $f_4$ ) The map

$$t \mapsto \frac{f(t)}{t^{2m-1}}$$

is increasing in  $|t| > 0$ .

The main result of this paper is:

**Theorem 1.1** *Suppose that  $(M_1) - (M_2)$ ,  $(\phi_1) - (\phi_3)$   $(f_1) - (f_4)$  hold. Then problem  $(P)$  possesses a least energy nodal solution, which has precisely two nodal domains.*

In the last years, many authors show the existence and/or multiplicity of nontrivial solutions for the problem  $(*)$ , as can be seen [4], [8], [14], [16], [17], [18], [23], [24], [25], [26], [32], [34], [35], [36], [37], [43], [46], [47], [48] and reference therein.

Only the articles [27], [40], [41], [44] and [50] consider solutions that change sign (nodal solution) for the Kirchhoff problem. In [40], [41], [44] and [50] the authors show the existence of solutions which change sign considering  $M(t) = a + bt$  for some positive constants  $a, b$ . In [50] the authors showed the existence of the sign changing solutions to the problem  $(*)$  for the cases 4-sublinear, asymptotically 4-linear and 4-superlinear. In that paper the authors

use variational methods and invariant sets of descent flow. In [40] the authors showed the same result found in [50] without considering the Ambrosetti-Rabinowitz condition. In [41] the authors study the result found in [50] considering now the case asymptotically 3-linear. In [27] and [44] the authors show the existence of nodal solution using a minimization argument and a quantitative deformation lemma. In [27] the function  $M$  was more general that included the case  $M(t) = a + bt$ .

The generalized Kirchhoff problem, that is, Kirchhoff problem in Orlicz-Sobolev spaces, was studied in [19], [28] and [29] using Genus theory in order to show a multiplicity result. In [20] the author used the Mountain Pass Theorem to show the existence result. In [21] the author used the Ricceri's three points critical result to show multiplicity result.

In this work we completes the results found in [19], [20], [21], [28], [29] and extend the studies found in [27], [40], [41], [44] and [50] in the following sense:

a) Unlike [19], [20], [21], [28], [29], we show the existence of a nodal solution for a generalized Kirchhoff problem.

b) Since we work in Orlicz-Sobolev spaces, some estimates were necessary, as for example, Lemma 3.4 and some results were more delicate, as for example, Lemma 3.5.

c) This result is new even in the case  $M \equiv 1$ .

d) Problem  $(P_\alpha)$  possesses more complicated nonlinearities, as for example:

(i)  $\Phi(t) = t^{p_0} + t^{p_1}$ ,  $1 < p_0 < p_1 < N$  and  $p_1 \in (p_0, p_0^*)$ , for  $N \geq 2$ .

(ii)  $\Phi(t) = (1 + t^2)^\gamma - 1$  for  $\gamma \in (1, 3)$  and  $N = 3$ .

(iii)  $\Phi(t) = t^p \log(t^q + 1)$ , with  $p \in (\frac{N}{2}, N)$ ,  $q \in (0, \min\{\frac{p}{N-p}, N-p\})$  and  $N \geq 3$ .

The paper is organized as follows. In the next section we give a brief review on Orlicz-Sobolev spaces. In section 3 we give the variational framework and we prove some technical lemmas. In the section 4 we prove Theorem 1.1.

## 2 A brief review on Orlicz-Sobolev spaces

Let  $\varphi$  be a real-valued function defined in  $[0, \infty)$  and having the following properties:

a)  $\varphi(0) = 0$ ,  $\varphi(t) > 0$  if  $t > 0$  and  $\lim_{t \rightarrow \infty} \varphi(t) = \infty$ .

b)  $\varphi$  is nondecreasing, that is,  $s > t$  implies  $\varphi(s) \geq \varphi(t)$ .

c)  $\varphi$  is right continuous, that is,  $\lim_{s \rightarrow t^+} \varphi(s) = \varphi(t)$ .

Then, the real-valued function  $\Phi$  defined on  $\mathbf{R}$  by

$$\Phi(t) = \int_0^{|t|} \varphi(s) \, ds$$

is called an N-function. For an N-function  $\Phi$  and an open set  $\Omega \subseteq \mathbf{R}^N$ , the Orlicz space  $L_\Phi(\Omega)$  is defined (see [1] or [31]). When  $\Phi$  satisfies  $\Delta_2$ -condition, that is, when there are

$t_0 \geq 0$  and  $K > 0$  such that  $\Phi(2t) \leq K\Phi(t)$ , for all  $t \geq t_0$ , the space  $L_\Phi(\Omega)$  is the vectorial space of the measurable functions  $u : \Omega \rightarrow \mathbf{R}$  such that

$$\int_{\Omega} \Phi(|u|) \, dx < \infty.$$

The space  $L_\Phi(\Omega)$  endowed with Luxemburg norm, that is, the norm given by

$$|u| = \inf \left\{ \tau > 0 : \int_{\Omega} \Phi\left(\frac{|u|}{\tau}\right) \, dx \leq 1 \right\},$$

is a Banach space. The complement function of  $\Phi$ , denoted by  $\tilde{\Phi}$ , is given by the Legendre transformation, that is

$$\tilde{\Phi}(s) = \max_{t \geq 0} \{st - \Phi(t)\} \quad \text{for } s \geq 0.$$

These  $\Phi$  and  $\tilde{\Phi}$  are complementary each other. Involving the functions  $\Phi$  and  $\tilde{\Phi}$ , we have the Young's inequality given by

$$st \leq \Phi(t) + \tilde{\Phi}(s).$$

Using the above inequality, it is possible to prove the following Hölder type inequality

$$\left| \int_{\Omega} uv \, dx \right| \leq 2|u|_{\Phi} |v|_{\tilde{\Phi}} \quad \forall u \in L_{\Phi}(\Omega) \text{ and } v \in L_{\tilde{\Phi}}(\Omega).$$

Hereafter, we denote by  $W_0^{\Phi}(\Omega)$  the Orlicz-Sobolev space obtained by the completion of  $C_0^{\infty}(\Omega)$  with norm

$$\|u\| = |u|_{\Phi} + |\nabla u|_{\Phi}.$$

When  $\Omega$  is bounded, for all  $u \in W_0^{\Phi}(\Omega)$ , there is  $c > 0$  such that

$$|u| \leq c|\nabla u|_{\Phi} \tag{2.1}$$

and

$$\int_{\Omega} \Phi(u) \, dx \leq c \int_{\Omega} \Phi(|\nabla u|) \, dx. \tag{2.2}$$

In this case, we can consider

$$\|u\| = |\nabla u|_{\Phi}.$$

Another important function related to function  $\Phi$ , is the Sobolev conjugate function  $\Phi_*$  of  $\Phi$  defined by

$$\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{(N+1)/N}} \, ds, \quad t > 0.$$

Another important inequality was proved by Donaldson and Trudinger [22], which establishes that for all open  $\Omega \subset \mathbf{R}^N$  and there is a constant  $S_N = S(N) > 0$  such that

$$|u|_{\Phi_*} \leq S_N |\nabla u|_{\Phi}, \quad \text{for all } u \in W_0^{1,\Phi}(\Omega). \tag{2.3}$$

This inequality shows the below embedding is continuous

$$W_0^{1,\Phi}(\Omega) \xrightarrow{\text{cont}} L_{\Phi_*}(\Omega).$$

If bounded domain  $\Omega$  and the limits below hold

$$\limsup_{t \rightarrow 0} \frac{B(t)}{\Phi(t)} < +\infty \text{ and } \limsup_{|t| \rightarrow +\infty} \frac{B(t)}{\Phi_*(t)} = 0, \quad (2.4)$$

the embedding

$$W_0^{1,\Phi}(\Omega) \hookrightarrow L_B(\Omega) \quad (2.5)$$

is compact.

The hypotheses  $(\phi_1) - (\phi_3)$  implies that  $\Phi$ ,  $\tilde{\Phi}$ ,  $\Phi_*$  and  $\tilde{\Phi}_*$  satisfy  $\Delta_2$ -condition. This condition allows us conclude that:

- 1)  $u_n \rightarrow 0$  in  $L_\Phi(\Omega)$  if, and only if,  $\int_\Omega \Phi(u_n) dx \rightarrow 0$ .
- 2)  $L_\Phi(\Omega)$  is separable and  $\overline{C_0^\infty(\Omega)}^{|\cdot|_\Phi} = L_\Phi(\Omega)$ .
- 3)  $L_\Phi(\Omega)$  is reflexive and its dual is  $L_{\tilde{\Phi}}(\Omega)$  (see [1]).

Under assumptions  $(\phi_1) - (\phi_3)$ , some elementary inequalities listed in the following lemmas are valid. For the proofs, see [30].

**Lemma 2.1** *Assume  $(\phi_1)$  and  $(\phi_2)$ . Then,*

$$\Phi(t) = \int_0^{|t|} s\phi(s)ds,$$

*is a  $N$ -function with  $\Phi, \tilde{\Phi} \in \Delta_2$ . Hence,  $L_\Phi(\Omega)$  and  $W_0^{1,\Phi}(\Omega)$  are reflexive and separable spaces.*

**Lemma 2.2** *The functions  $\Phi$ ,  $\Phi_*$ ,  $\tilde{\Phi}$  and  $\tilde{\Phi}_*$  satisfy the inequalities*

$$\tilde{\Phi}(\phi(t)t) \leq \Phi(2t) \text{ and } \tilde{\Phi}\left(\frac{\Phi(t)}{t}\right) \leq \Phi(t) \quad \forall t \geq 0. \quad (2.6)$$

**Lemma 2.3** *Assume  $(\phi_1)$  and  $(\phi_2)$  hold and let  $\xi_0(t) = \min\{t^l, t^m\}$ ,  $\xi_1(t) = \max\{t^l, t^m\}$ , for all  $t \geq 0$ . Then,*

$$\xi_0(\rho)\Phi(t) \leq \Phi(\rho t) \leq \xi_1(\rho)\Phi(t) \text{ for } \rho, t \geq 0$$

*and*

$$\xi_0(|u|_\Phi) \leq \int_\Omega \Phi(u)dx \leq \xi_1(|u|_\Phi) \text{ for } u \in L_\Phi(\Omega).$$

**Lemma 2.4** *Assume  $(\phi_3)$  holds and let  $\Psi(t) = \phi(t)t$ ,  $\tau_0(t) = \min\{t^{l-1}, t^{m-1}\}$ ,  $\tau_1(t) = \max\{t^{l-1}, t^{m-1}\}$ , for all  $t \geq 0$ . Then,*

$$\tau_0(\rho)\Psi(t) \leq \Psi(\rho t) \leq \tau_1(\rho)\Psi(t) \text{ for } \rho, t \geq 0.$$

**Lemma 2.5** *The function  $\Phi_*$  satisfies the following inequality*

$$l^* \leq \frac{\Phi'_*(t)t}{\Phi_*(t)} \leq m^* \text{ for } t > 0. \quad (2.7)$$

As an immediate consequence of the Lemma 2.5, we have the following result:

**Lemma 2.6** *Assume  $(\phi_1) - (\phi_2)$  hold and let  $\xi_2(t) = \min\{t^{l^*}, t^{m^*}\}$ ,  $\xi_3(t) = \max\{t^{l^*}, t^{m^*}\}$  for all  $t \geq 0$ . Then,*

$$\xi_2(\rho)\Phi_*(t) \leq \Phi_*(\rho t) \leq \xi_3(\rho)\Phi_*(t) \text{ for } \rho, t \geq 0$$

and

$$\xi_2(|u|_{\Phi_*}) \leq \int_{\Omega} \Phi_*(u) dx \leq \xi_3(|u|_{\Phi_*}) \text{ for } u \in L_{\Phi_*}(\Omega).$$

**Lemma 2.7** *Let  $\tilde{\Phi}$  be the complement of  $\Phi$  and put*

$$\xi_4(s) = \min\{s^{\frac{l}{l-1}}, s^{\frac{m}{m-1}}\} \text{ and } \xi_5(s) = \max\{s^{\frac{l}{l-1}}, s^{\frac{m}{m-1}}\}, \quad s \geq 0.$$

*Then the following inequalities hold*

$$\xi_4(r)\tilde{\Phi}(s) \leq \tilde{\Phi}(rs) \leq \xi_5(r)\tilde{\Phi}(s), \quad r, s \geq 0$$

and

$$\xi_4(|u|_{\tilde{\Phi}}) \leq \int_{\Omega} \tilde{\Phi}(u) dx \leq \xi_5(|u|_{\tilde{\Phi}}), \quad u \in L_{\tilde{\Phi}}(\Omega).$$

### 3 Variational framework and technical lemmas

We say that  $u \in W_0^{1,\Phi}(\Omega)$  is a weak nodal solution of the problem (P) if  $u^+ \neq 0$ ,  $u^- \neq 0$  in  $\Omega$  and it verifies

$$M\left(\int_{\Omega} \Phi(|\nabla u|) dx\right) \int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v dx - \int_{\Omega} f(u)v dx = 0, \text{ for all } v \in W_0^{1,\Phi}(\Omega).$$

In view of  $(f_1) - (f_2)$ , we have that the functional  $J : W_0^{1,\Phi}(\Omega) \rightarrow \mathbf{R}$  given by

$$J(u) := \widehat{M}\left(\int_{\Omega} \Phi(|\nabla u|) dx\right) - \int_{\Omega} F(u) dx$$

is well defined, where  $\widehat{M}(t) = \int_0^t M(s) ds$ . Moreover,  $J \in C^1(W_0^{1,\Phi}(\Omega), \mathbf{R})$  with the following derivative

$$J'(u)v = M\left(\int_{\Omega} \Phi(|\nabla u|) dx\right) \int_{\Omega} \phi(|\nabla u|) \nabla u \nabla v dx - \int_{\Omega} f(u)v dx,$$

for all  $v \in W_0^{1,\Phi}(\Omega)$ . Thus, the weak solutions of  $(P)$  are precisely the critical points of  $J$ . Associated to the functional  $J$  we define the Nehari manifold

$$\mathcal{N} := \left\{ u \in W_0^{1,\Phi}(\Omega) \setminus \{0\} : J'(u)u = 0 \right\}.$$

In the Theorem 1.1 we prove that there is  $w \in \mathcal{M}$  such that

$$J(w) = \min_{v \in \mathcal{M}} J(v),$$

where

$$\mathcal{M} := \left\{ w \in \mathcal{N} : J'(w)w^+ = 0 = J'(w)w^- \right\}.$$

From  $(M_1)$  we have that  $M \left( \int_{\Omega} \Phi(|\nabla w^{\pm}|) dx \right) \leq M \left( \int_{\Omega} \Phi(|\nabla w|) dx \right)$ , for  $w \in \mathcal{M}$ . Thus, this last inequality implies that

$$J'(w^{\pm})w^{\pm} \leq 0, \quad \text{for all } w \in \mathcal{M}. \quad (3.1)$$

Let us begin by establishing some preliminary results which will be exploited in the last section for a minimization argument.

**Lemma 3.1 (a)** *For all  $u \in \mathcal{N}$  we have*

$$J(u) \geq \frac{(\theta - 4m)}{4\theta} \sigma_0 \xi_0 (|\nabla u|_{\Phi}).$$

**(b)** *There is  $\rho > 0$  such that*

$$\|u\| \geq \rho, \quad \text{for all } u \in \mathcal{N}$$

*and*

$$\|w^{\pm}\| \geq \rho, \quad \text{for all } w \in \mathcal{M}.$$

**Proof.** From definition of  $\widehat{M}$  and  $(M_2)$ , we get

$$\widehat{M}(t) \geq \frac{1}{2} M(t)t, \quad \text{for all } t \geq 0. \quad (3.2)$$

Thus, by  $(f_3)$  we get

$$J(u) = J(u) - \frac{1}{\theta} J'(u)u \geq \widehat{M} \left( \int_{\Omega} \Phi(|\nabla u|) dx \right) - \frac{1}{\theta} M \left( \int_{\Omega} \Phi(|\nabla u|) dx \right) \int_{\Omega} \phi(|\nabla u|) |\nabla u|^2 dx.$$

Using  $(\phi_2)$  we obtain

$$J(u) \geq \widehat{M} \left( \int_{\Omega} \Phi(|\nabla u|) dx \right) - \frac{m}{\theta} M \left( \int_{\Omega} \Phi(|\nabla u|) dx \right) \int_{\Omega} \Phi(|\nabla u|) dx.$$

Since  $\theta \in (2m, l^*)$ , by (3.2) and Lemma 2.3, the proof of (a) is finished.



To prove (b), notice that for all  $u \in W_0^{1,\Phi}(\Omega)$ , by  $(f_1)$ ,  $(f_2)$ ,  $(\phi_2)$ , (2.2) and (2.7), given  $\epsilon > 0$ , there exists  $C_\epsilon > 0$  such that

$$\int_{\Omega} f(u)u dx \leq \epsilon cm \int_{\Omega} \Phi(|\nabla u|) dx + C_\epsilon cm^* \int_{\Omega} \Phi_*(u) dx. \quad (3.3)$$

Now from definition of  $\mathcal{N}$ ,  $(M_1)$ ,  $(\phi_2)$  again and (3.3), we have

$$(\sigma_0 l - \epsilon cm) \int_{\Omega} \Phi(|\nabla u|) dx \leq C_\epsilon cm^* \int_{\Omega} \Phi_*(u) dx.$$

By Lemma 2.3 and (2.3) we obtain

$$(\sigma_0 l - \epsilon cm) \xi_0(|\nabla u|_{\Phi}) \leq c C C_\epsilon m \xi_3(|\nabla u|_{\Phi}).$$

If  $|\nabla u|_{\Phi} \geq 1$ , the proof is done. Suppose that  $|\nabla u|_{\Phi} \leq 1$ . Then, from the last inequality we have

$$(\sigma_0 l - \epsilon cm) |\nabla u|_{\Phi}^m \leq c C C_\epsilon m |\nabla u|_{\Phi}^{l^*}.$$

Consequently,

$$\left( \frac{\sigma_0 l - \epsilon cm}{c C C_\epsilon m} \right)^{\frac{1}{(l^* - m)}} \leq |\nabla u|_{\Phi},$$

for all  $u \in \mathcal{N}$ . Then proof of (b) is finished with  $\rho = \min \left\{ 1, \left( \frac{\sigma_0 l - \epsilon cm}{c C C_\epsilon m} \right)^{\frac{1}{(l^* - m)}} \right\}$ .

From (3.1) and repeating the reasoning before we obtain

$$0 < \rho \leq \|w^\pm\|. \blacksquare$$

We apply the next result in the last section to every bounded minimizing sequence of  $J$  on  $\mathcal{M}$  in order to ensure that the candidate minimizer is different from zero.

**Lemma 3.2** *If  $(w_n)$  is a bounded sequence in  $\mathcal{M}$ , then there is a positive constant  $q \in (m, l^*)$  such that*

$$\liminf_{n \rightarrow \infty} \int_{\Omega} |w_n^\pm|^q dx > 0.$$

**Proof.** Note that by  $(f_1)$ ,  $(f_2)$ ,  $(\phi_2)$  and (2.7), given  $\epsilon > 0$ , there exists  $C_\epsilon > 0$  such that

$$\int_{\Omega} f(w_n^\pm) w_n^\pm dx \leq \epsilon m \int_{\Omega} \Phi(|\nabla w_n^\pm|) dx + \epsilon m^* \int_{\Omega} \Phi_*(w_n^\pm) dx + C_\epsilon \int_{\Omega} |w_n^\pm|^q dx. \quad (3.4)$$

On the other hand, by  $(M_1)$ ,  $(\phi_2)$  and Lemma 2.3, we get

$$\sigma_0 l \xi_0(|\nabla w_n^\pm|_{\Phi}) \leq l \sigma_0 \int_{\Omega} \Phi(|\nabla w_n^\pm|) dx \leq \int_{\Omega} f(w_n^\pm) w_n^\pm dx. \quad (3.5)$$

Using (3.4), (3.5) and last Lemma, we obtain

$$0 < \sigma_0 \xi_0(\rho) \leq \epsilon m \int_{\Omega} \Phi(|\nabla w_n^\pm|) dx + \epsilon m^* \int_{\Omega} \Phi_*(w_n^\pm) dx + C_\epsilon \int_{\Omega} |w_n^\pm|^q dx.$$

Since  $(w_n)$  is bounded in  $W_0^{1,\Phi}(\Omega)$ , by Lemma 2.3, there is  $C > 0$  such that

$$0 < \sigma_0 l \xi_0(\rho) \leq \epsilon C + C_\epsilon \int_\Omega |w_n^\pm|^q dx$$

and the result follows of the last inequality.  $\blacksquare$

Next results try to infer geometrical information of  $J$  with respect to  $\mathcal{M}$  in the same way that one is used to do about  $\mathcal{N}$ . To be more precise, note the similarity between the next result and that which states that for each  $v \in W_0^{1,\Phi}(\Omega) \setminus \{0\}$  there exists  $t_v > 0$  such that  $t_v v \in \mathcal{N}$ .

**Lemma 3.3** *If  $v \in W_0^{1,\Phi}(\Omega)$  with  $v^\pm \neq 0$ , then there are  $t, s > 0$  such that*

$$J'(tv^+ + sv^-)v^+ = 0$$

and

$$J'(tv^+ + sv^-)v^- = 0.$$

**Proof.** Let  $V : (0, +\infty) \times (0, +\infty) \rightarrow \mathbf{R}^2$  be a continuous function given by

$$V(t, s) = (J'(tv^+ + sv^-)(tv^+), J'(tv^+ + sv^-)(sv^-)).$$

Note that

$$\begin{aligned} J'(tv^+ + sv^-)(tv^+) &= M \left( \int_\Omega \Phi(|\nabla(tv^+ + sv^-)|) dx \right) \int_\Omega \phi(|\nabla(tv^+)|) |\nabla(tv^+)|^2 dx \\ &\quad - \int_\Omega f(tv^+) tv^+ dx. \end{aligned} \quad (3.6)$$

Using  $(M_1)$  and  $(\phi_2)$  we have

$$J'(tv^+ + sv^-)(tv^+) \geq \sigma_0 l \int_\Omega \Phi(|\nabla(tv^+)|) dx - \int_\Omega f(tv^+) tv^+ dx.$$

Considering (3.3) in the last inequality we get

$$J'(tv^+ + sv^-)(tv^+) \geq \sigma_0 l \int_\Omega \Phi(|\nabla(tv^+)|) dx - \epsilon \int_\Omega \Phi(tv^+) dx - C_\epsilon \int_\Omega \Phi_*(tv^+) dx.$$

From (2.2) we obtain

$$J'(tv^+ + sv^-)(tv^+) \geq (\sigma_0 l - \epsilon c) \int_\Omega \Phi(|\nabla(tv^+)|) dx - C_\epsilon \int_\Omega \Phi_*(tv^+) dx.$$

Now, by Lemma 2.3 we derive

$$J'(tv^+ + sv^-)(tv^+) \geq (\sigma_0 l - \epsilon c) \xi_0(t) \xi_0(|\nabla(v^+)|_\Phi) - C_\epsilon \xi_3(t) \xi_3(|\nabla(v^+)|_\Phi).$$

Thus, there exists  $r > 0$  sufficiently small such that

$$J'(rv^+ + sv^-)(rv^+) \geq (\sigma_0 l - \epsilon c) r^m \xi_0(|\nabla(v^+)|_\Phi) - C_\epsilon r^{l^*} \xi_3(|\nabla(v^+)|_\Phi) > 0, \quad \text{for all } s > 0.$$

Arguing of the same way we get

$$J'(tv^+ + rv^-)(rv^-) > 0,$$

for all  $t > 0$  and  $r > 0$  sufficiently small.

On the other hand, using  $(\phi_2)$  in (3.6) we get

$$\begin{aligned} J'(tv^+ + sv^-)(tv^+) &\leq mM \left( \int_{\Omega} \Phi(|\nabla(tv^+)|) dx \right) \int_{\Omega} \Phi(|\nabla(tv^+ + sv^-)|) dx \\ &\quad - \int_{\Omega} f(tv^+) tv^+ dx. \end{aligned} \quad (3.7)$$

Note that, by  $(M_2)$ , there exists  $K_1 > 0$  such that

$$M(t) \leq M(1)t + K_1, \quad \text{for all } t \geq 0. \quad (3.8)$$

Using (3.8) in (3.7) and recalling that  $v^+$  and  $v^-$  have compact support disjoint we obtain

$$\begin{aligned} J'(tv^+ + sv^-)(tv^+) &\leq mM(1) \left( \int_{\Omega} \Phi(|\nabla(tv^+)|) dx \right)^2 \\ &\quad + mM(1) \left( \int_{\Omega} \Phi(|\nabla(tv^+)|) dx \right) \left( \int_{\Omega} \Phi(|\nabla(sv^-)|) dx \right) \\ &\quad + mK_1 \int_{\Omega} \Phi(|\nabla(tv^+)|) dx - \int_{\Omega} f(tv^+) tv^+ dx. \end{aligned}$$

By Lemma 2.3 and  $(f_3)$  we have

$$\begin{aligned} J'(tv^+ + sv^-)(tv^+) &\leq mM(1)\xi_1(t)^2\xi_1(|\nabla(v^+)|_{\Phi})^2 \\ &\quad + mM(1)\xi_1(t)\xi_1(s)\xi_1(|\nabla(tv^+)|_{\Phi})\xi_1(|\nabla(v^-)|_{\Phi}) \\ &\quad + mK_1\xi_1(t)\xi_1(|\nabla(v^+)|_{\Phi}) - \theta \int_{\Omega} F(tv^+) dx. \end{aligned} \quad (3.9)$$

Note that, by  $(f_3)$  again, there are  $K_2, K_3 > 0$  such that

$$F(t) \geq K_2 t^{\theta} - K_3. \quad (3.10)$$

Using (3.10) in (3.9), we have

$$\begin{aligned} J'(tv^+ + sv^-)(tv^+) &\leq mM(1)\xi_1(t)^2\xi_1(|\nabla(v^+)|_{\Phi})^2 \\ &\quad + mM(1)\xi_1(t)\xi_1(s)\xi_1(|\nabla(tv^+)|_{\Phi})\xi_1(|\nabla(v^-)|_{\Phi}) \\ &\quad + mK_1\xi_1(t)\xi_1(|\nabla(v^+)|_{\Phi}) - \theta t^{\theta} K_2 |v^+|_{\theta}^{\theta} + K_3 \theta |\Omega|, \end{aligned}$$

where  $|\Omega|$  denotes the Lebesgue measure of  $\Omega$ . Thus, since  $\theta > 2m$ , for  $s \leq t \leq R$  and  $R > 0$  sufficiently large, we get

$$\begin{aligned} J'(tv^+ + sv^-)(tv^+) &\leq mM(1)R^{2m}\xi_1(|\nabla(v^+)|_\Phi)^2 \\ &\quad + mM(1)R^{2m}\xi_1(|\nabla(tv^+)|_\Phi)\xi_1(|\nabla(v^-)|_\Phi) \\ &\quad + mK_1R^m\xi_1(|\nabla(v^+)|_\Phi) - \theta K_2R^\theta|v^+|_\theta^\theta + K_3\theta|\Omega| < 0. \end{aligned}$$

Arguing of the same way we get

$$J'(tv^+ + Rv^-)(Rv^-) < 0, \text{ for all } t \leq R.$$

In particular,

$$J'(rv^+ + sv^-)(rv^+) > 0 \text{ and } J'(tv^+ + rv^-)(rv^-) > 0, \text{ for all } t, s \in [r, R]$$

and

$$J'(Rv^+ + sv^-)(Rv^+) < 0 \text{ and } J'(tv^+ + Rv^-)(Rv^-) < 0, \text{ for all } t, s \in [r, R].$$

Now the lemma follows applying Miranda's theorem [42]. ■

In the next Lemma we prove monotonicity results for some functions that will be much useful in our arguments.

**Lemma 3.4**

(a) *The function*

$$t \mapsto \widehat{M}(t) - \frac{1}{2}M(t)t \text{ is increasing.} \quad (3.11)$$

(b) *For all  $v \in W_0^{1,\Phi}(\Omega)$  with  $v \geq 0$  and  $v \neq 0$ , the function*

$$t \mapsto \widehat{M}\left(\int_\Omega \Phi(tv)dx\right) - \frac{1}{2m}M\left(\int_\Omega \Phi(tv)dx\right)\int_\Omega \phi(tv)(tv)^2dx \text{ is increasing.} \quad (3.12)$$

(c) *The function*

$$t \mapsto \frac{1}{2m}f(t)t - F(t) \text{ is increasing, for all } |t| > 0. \quad (3.13)$$

(d)

$$\widehat{M}(t+s) \geq \widehat{M}(t) + \widehat{M}(s), \text{ for all } t, s \in [0, +\infty). \quad (3.14)$$

**Proof.** (a) First of all, let us observe that, from  $(M_2)$  we have

$$M'(t)t \leq M(t), \text{ for all } t \geq 0. \quad (3.15)$$

Now from (3.15) we conclude  $(\widehat{M}(t) - \frac{1}{2}M(t)t)' > 0$  which implies

$$t \mapsto \widehat{M}(t) - \frac{1}{2}M(t)t \text{ is increasing.}$$

Now we prove (b). For each  $v \in W_0^{1,\Phi}(\Omega)$  with  $v \geq 0$  and  $v \neq 0$ , we define the function

$$\zeta(t) = \widehat{M}\left(\int_{\Omega} \Phi(tv)dx\right) - \frac{1}{2m}M\left(\int_{\Omega} \Phi(tv)dx\right) \int_{\Omega} \phi(tv)(tv)^2 dx.$$

Then, recalling that  $\Psi(t) = \phi(t)t$  ( see Lemma 2.4 ), we have

$$\begin{aligned} \zeta'(t) &= M\left(\int_{\Omega} \Phi(tv)dx\right) \int_{\Omega} \Psi(tv)v dx \\ &\quad - \frac{1}{2m}M'\left(\int_{\Omega} \Phi(tv)dx\right) \int_{\Omega} \Psi(tv)v dx \int_{\Omega} \Psi(tv)tv dx \\ &\quad - \frac{1}{2m}M\left(\int_{\Omega} \Phi(tv)dx\right) \int_{\Omega} (\Psi'(tv)tv^2 + \Psi(tv)v) dx. \end{aligned}$$

Now using  $(\phi_3)$  we have

$$\begin{aligned} \zeta'(t) &\geq \int_{\Omega} \Psi(tv)v dx \left( M\left(\int_{\Omega} \Phi(tv)dx\right) \right. \\ &\quad \left. - \frac{1}{2m}M'\left(\int_{\Omega} \Phi(tv)dx\right) \int_{\Omega} \Psi(tv)tv dx \right. \\ &\quad \left. - \frac{m-1}{2m}M\left(\int_{\Omega} \Phi(tv)dx\right) - \frac{1}{2m}M\left(\int_{\Omega} \Phi(tv)dx\right) \right). \end{aligned}$$

Using  $(\phi_2)$  we get

$$\zeta'(t) \geq \int_{\Omega} \Psi(tv)v dx \left( \frac{1}{2}M\left(\int_{\Omega} \Phi(tv)dx\right) - \frac{1}{2}M'\left(\int_{\Omega} \Phi(tv)dx\right) \int_{\Omega} \Phi(tv)dx \right).$$

By (3.15) we obtain  $\zeta'(t) > 0$  and the proof of (b) is done.

Now we prove (c). Using  $(f_4)$  we get

$$f'(t)t \geq (2m-1)f(t), \text{ for all } |t| \geq 0, \quad (3.16)$$

that implies

$$\left(\frac{1}{2m}f(t)t - F(t)\right)' > 0,$$

that is,

$$t \mapsto \frac{1}{2m}f(t)t - F(t) \text{ is increasing, for all } |t| > 0.$$

Finally, we prove (d). From  $(M_1)$  we obtain

$$\begin{aligned}
\widehat{M}(t+s) &= \int_0^{t+s} M(\tau) d\tau = \widehat{M}(t) + \int_t^{t+s} M(\tau) d\tau \\
&= \widehat{M}(t) + \int_0^s M(\gamma+t) d\gamma \\
&\geq \widehat{M}(t) + \int_0^s M(\gamma) d\gamma \\
&= \widehat{M}(t) + \widehat{M}(s), \quad \text{for all } t, s \in [0, +\infty). \blacksquare
\end{aligned}$$

Now, we can define a suitable function and its gradient vector field which are related to functional  $J$  and will be involved in particular in the application of the deformation lemma. Indeed, for each  $v \in W_0^{1,\Phi}(\Omega)$  with  $v^\pm \neq 0$  we consider

$$h^v : [0, +\infty) \times [0, +\infty) \rightarrow \mathbf{R} \quad \text{given by} \quad h^v(t, s) = J(tv^+ + sv^-)$$

and its gradient  $\Upsilon^v : [0, +\infty) \times [0, +\infty) \rightarrow \mathbf{R}^2$  defined by

$$\begin{aligned}
\Upsilon^v(t, s) &= \left( \Upsilon_1^v(t, s), \Upsilon_2^v(t, s) \right) = \left( \frac{\partial h^v}{\partial t}(t, s), \frac{\partial h^v}{\partial s}(t, s) \right) \\
&= (J'(tv^+ + sv^-)v^+, J'(tv^+ + sv^-)v^-),
\end{aligned}$$

for every  $(t, s) \in [0, +\infty) \times [0, +\infty)$ . Furthermore, we consider the Hessian matrix of  $h^v$  or the Jacobian matrix of  $\Upsilon^v$ , i.e.

$$(\Upsilon^v)'(t, s) = \begin{pmatrix} \frac{\partial \Upsilon_1^v}{\partial t}(t, s) & \frac{\partial \Upsilon_1^v}{\partial s}(t, s) \\ \frac{\partial \Upsilon_2^v}{\partial t}(t, s) & \frac{\partial \Upsilon_2^v}{\partial s}(t, s) \end{pmatrix},$$

for every  $(t, s) \in [0, +\infty) \times [0, +\infty)$ . Indeed, in the following we aim to prove that, if  $w \in \mathcal{M}$ , function  $h^w$  has a critical point and in particular a global maximum in  $(t, s) = (1, 1)$ ,

**Lemma 3.5** *If  $w \in \mathcal{M}$ , then*

(a)

$$h^w(t, s) < h^w(1, 1) = J(w),$$

for all  $t, s \geq 0$  such that  $(t, s) \neq (1, 1)$ .

(b)

$$\det(\Upsilon^w)'(1, 1) > 0.$$

**Proof.** Since  $w \in \mathcal{M}$ , then

$$J'(w)w^\pm = J'(w^+ + w^-)w^\pm = 0.$$

Thus,

$$\Upsilon^w(1, 1) = \left( \frac{\partial h^w}{\partial t}(1, 1), \frac{\partial h^w}{\partial s}(1, 1) \right) = (0, 0).$$

Moreover, from (3.8) and (3.10), for  $t$  and  $s$  sufficiently large, we get

$$\begin{aligned} h^w(t, s) = J(tw^+ + sw^-) &\leq mM(1)t^{2m}\xi_1(|\nabla(w^+)|_\Phi)^2 + mM(1)s^{2m}\xi_1(|\nabla(w^-)|_\Phi)^2 \\ &\quad + mM(1)t^{2m}\xi_1(|\nabla(tw^+)|_\Phi)\xi_1(|\nabla(w^-)|_\Phi) \\ &\quad + mM(1)s^{2m}\xi_1(|\nabla(sw^-)|_\Phi)\xi_1(|\nabla(w^+)|_\Phi) \\ &\quad + mK_1t^m\xi_1(|\nabla(w^+)|_\Phi) + mK_1s^m\xi_1(|\nabla(w^-)|_\Phi) \\ &\quad - K_2\theta t^\theta|w^+|_\theta^\theta + 2K_3\theta|\Omega| - K_2\theta s^\theta|w^-|_\theta^\theta. \end{aligned}$$

Since  $2m < \theta$ , then

$$\lim_{|(t,s)| \rightarrow +\infty} h^w(t, s) = -\infty,$$

that implies  $(1, 1)$  is a critical point of  $h^w$  and  $h^w$  has a global maximum point in  $(a, b)$ .

Now we prove that  $a, b > 0$ . Suppose, by contradiction that  $b = 0$ . Thus,

$$J'(aw^+)aw^+ = 0$$

implies

$$M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right) \int_{\Omega} \phi(a|\nabla w^+|)a^2|\nabla w^+|^2dx = \int_{\Omega} f(aw^+)aw^+dx.$$

Then, by Lemma 2.4, we have

$$M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right) \tau_1(a)a \int_{\Omega} \phi(|\nabla w^+|)|\nabla w^+|^2dx \geq \int_{\Omega} f(aw^+)aw^+dx,$$

which is equivalent

$$M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right) \int_{\Omega} \phi(|\nabla w^+|)|\nabla w^+|^2dx \geq \int_{\Omega} \frac{f(aw^+)aw^+dx}{\tau_1(a)a}.$$

Hence,

$$\frac{M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right)}{\int_{\Omega} \Phi(a|\nabla w^+|)dx} \int_{\Omega} \Phi(a|\nabla w^+|)dx \int_{\Omega} \phi(|\nabla w^+|)|\nabla w^+|^2dx \geq \int_{\Omega} \frac{f(aw^+)aw^+dx}{\tau_1(a)a}.$$

Since  $\xi_1(a) = \tau_1(a)a$  we have

$$\frac{M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right)}{\int_{\Omega} \Phi(a|\nabla w^+|)dx} \int_{\Omega} \Phi(|\nabla w^+|)dx \int_{\Omega} \phi(|\nabla w^+|)|\nabla w^+|^2dx \geq \int_{\Omega} \frac{f(aw^+)aw^+dx}{\xi_1(a)^2}. \quad (3.17)$$

On the other hand, since  $J(w)w^+ = 0$  and  $M$  is increasing, we get

$$\frac{M\left(\int_{\Omega} \Phi(|\nabla w^+|)dx\right)}{\int_{\Omega} \Phi(|\nabla w^+|)dx} \int_{\Omega} \Phi(|\nabla w^+|)dx \int_{\Omega} \phi(|\nabla w^+|)|\nabla w^+|^2dx \leq \int_{\Omega} f(w^+)w^+dx. \quad (3.18)$$

Considering (3.17) and (3.18) we have

$$\begin{aligned} & \left[ \frac{M\left(\int_{\Omega} \Phi(|\nabla w^+|)dx\right)}{\int_{\Omega} \Phi(|\nabla w^+|)dx} - \frac{M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right)}{\int_{\Omega} \Phi(a|\nabla w^+|)dx} \right] \int_{\Omega} \Phi(|\nabla w^+|)dx \int_{\Omega} \phi(|\nabla w^+|)|\nabla w^+|^2 dx \\ & \leq \int_{\Omega} \left[ \frac{f(w^+)}{(w^+)^{2m-1}} - \frac{f(aw^+)a}{\xi_1(a)^2(w^+)^{2m-1}} \right] (w^+)^{2m} dx. \end{aligned}$$

The last inequality,  $(M_2)$  and  $(f_4)$  imply that  $a \leq 1$ , because otherwise we get

$$\begin{aligned} 0 & < \left[ \frac{M\left(\int_{\Omega} \Phi(|\nabla w^+|)dx\right)}{\int_{\Omega} \Phi(|\nabla w^+|)dx} - \frac{M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right)}{\int_{\Omega} \Phi(a|\nabla w^+|)dx} \right] \int_{\Omega} \Phi(|\nabla w^+|)dx \int_{\Omega} \phi(|\nabla w^+|)|\nabla w^+|^2 dx \\ & \leq \int_{\Omega} \left[ \frac{f(w^+)}{(w^+)^{2m-1}} - \frac{f(aw^+)}{(aw^+)^{2m-1}} \right] (w^+)^{2m} dx < 0. \end{aligned}$$

Now note that

$$\begin{aligned} h^w(a, 0) &= J(aw^+) = J(aw^+) - \frac{1}{2m} J'(aw^+)(aw^+) \\ &= \left[ \widehat{M}\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right) - \frac{1}{2m} M\left(\int_{\Omega} \Phi(a|\nabla w^+|)dx\right) \int_{\Omega} \phi(a|\nabla w^+|)|aw^+|^2 dx \right] \\ &+ \int_{\Omega} \left[ \frac{1}{2m} f(aw^+)aw^+ - F(aw^+) \right] dx. \end{aligned} \quad (3.19)$$

Since  $0 < a \leq 1$  and using (3.12) and (3.13) in (3.19) we obtain

$$\begin{aligned} h^w(a, 0) &\leq \left[ \widehat{M}\left(\int_{\Omega} \Phi(|\nabla w^+|)dx\right) - \frac{1}{2m} M\left(\int_{\Omega} \Phi(|\nabla w^+|)dx\right) \int_{\Omega} \phi(|\nabla w^+|)|w^+|^2 dx \right] \\ &+ \int_{\Omega} \left[ \frac{1}{2m} f(w^+)w^+ - F(w^+) \right] dx \\ &= J(w^+) - \frac{1}{2m} J'(w^+)w = J(w^+) = h^w(1, 0). \end{aligned}$$

Now, our aim is to prove that

$$J(w^+) = h^w(1, 0) < J(w) = h^w(1, 1).$$

By Lemma 3.1 we have that  $J(w^-) \geq 0$ . Thus,

$$\begin{aligned} J(w^+) &\leq J(w^+) + J(w^-) = \widehat{M}\left(\int_{\Omega} \Phi(|\nabla w^+|)dx\right) + \widehat{M}\left(\int_{\Omega} \Phi(|\nabla w^-|)dx\right) \\ &- \int_{\Omega} (F(w^+) + F(w^-)) dx. \end{aligned} \quad (3.20)$$



By (3.14) we get

$$J(w^+) < \widehat{M} \left( \int_{\Omega} \Phi(|\nabla w^+|) dx + \int_{\Omega} \Phi(|\nabla w^-|) dx \right) - \int_{\Omega} (F(w^+) + F(w^-)) dx.$$

Since the supports of  $w^+$  and  $w^-$  are disjoint, we obtain

$$h^w(1, 0) = J(w^+) < J(w) = h^w(1, 1),$$

which is an absurd because  $(a, 0)$  is a maximum point. The same way we prove that  $0 < a$ .

Now we will prove that  $0 < a, b \leq 1$ . Since  $(a, b)$  is another critical point of  $h^w$ , we have

$$M \left( \int_{\Omega} \Phi(a|\nabla w^+|) dx + \int_{\Omega} \Phi(b|\nabla w^-|) dx \right) \int_{\Omega} \phi(a|\nabla w^+|) |aw^+|^2 dx = \int_{\Omega} f(aw^+) aw^+ dx.$$

Without loss of generality, we can suppose that  $b \leq a$ . Suppose by contradiction  $a \geq 1$ . Thus

$$\int_{\Omega} f(aw^+) aw^+ dx \leq \frac{M \left( \int_{\Omega} \Phi(a|\nabla w|) dx \right)}{\int_{\Omega} \Phi(a|\nabla w|) dx} \int_{\Omega} \Phi(a|\nabla w|) dx \int_{\Omega} \phi(a|\nabla w^+|) |aw^+|^2 dx.$$

Then,

$$\int_{\Omega} \frac{f(aw^+)}{(aw^+)^{2m-1}} (w^+)^{2m} dx \leq \frac{M \left( \int_{\Omega} \Phi(a|\nabla w|) dx \right)}{\int_{\Omega} \Phi(a|\nabla w|) dx} \int_{\Omega} \Phi(a|\nabla w|) dx \int_{\Omega} \phi(a|\nabla w^+|) |w^+|^2 dx \quad (3.21)$$

On the other hand,  $J'(w)w^+ = 0$  implies

$$\frac{M \left( \int_{\Omega} \Phi(|\nabla w|) dx \right)}{\int_{\Omega} \Phi(|\nabla w|) dx} \int_{\Omega} \Phi(|\nabla w|) dx \int_{\Omega} \phi(|\nabla w^+|) |w^+|^2 dx = \int_{\Omega} f(w^+) w^+ dx. \quad (3.22)$$

Combining (3.21) and (3.22) and  $(M_2)$  and  $(f_4)$ , we get

$$\begin{aligned} 0 &< \left[ \frac{M \left( \int_{\Omega} \Phi(|\nabla w|) dx \right)}{\int_{\Omega} \Phi(|\nabla w|) dx} - \frac{M \left( \int_{\Omega} \Phi(a|\nabla w|) dx \right)}{\int_{\Omega} \Phi(a|\nabla w|) dx} \right] \int_{\Omega} \Phi(|\nabla w|) dx \int_{\Omega} \phi(|\nabla w^+|) |w^+|^2 dx \\ &\leq \int_{\Omega} \left[ \frac{f(w^+)}{(w^+)^{2m-1}} - \frac{f(aw^+)}{(aw^+)^{2m-1}} \right] (w^+)^{2m} dx < 0, \end{aligned} \quad (3.23)$$

which is a contradiction. Then,  $0 < b \leq a < 1$ .

Now we will prove that  $h^w$  does not have global maximum in  $[0, 1] \times [0, 1] \setminus \{(1, 1)\}$ . We will show that

$$h^w(a, b) < h^w(1, 1).$$

Note that  $|\nabla(aw^+ + bw^-)| = a|\nabla w^+| + b|\nabla w^-| \leq |\nabla w^+| + |\nabla w^-|$  and since  $\widehat{M}$  and  $\Phi$  are increasing we have

$$h^w(a, b) = J(aw^+ + bw^-) \leq \widehat{M} \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) - \int_{\Omega} F(aw^+ + bw^-) dx. \quad (3.24)$$

By  $(f_3)$  we get  $\frac{1}{2m} \int_{\Omega} f(aw^+ + bw^-)(aw^+ + bw^-) dx \geq 0$ . Thus, put this information in (3.24) we obtain

$$\begin{aligned} h^w(a, b) &= J(aw^+ + bw^-) \leq \widehat{M} \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \\ &+ \int_{\Omega} \left[ \frac{1}{2m} f(aw^+ + bw^-)(aw^+ + bw^-) - F(aw^+ + bw^-) \right] dx. \end{aligned}$$

Since  $w^+$  and  $w^-$  have supports disjoint we get

$$\begin{aligned} h^w(a, b) &\leq \widehat{M} \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \\ &+ \int_{\Omega} \left[ \frac{1}{2m} f(aw^+)(aw^+) - F(aw^+) \right] dx \\ &+ \int_{\Omega} \left[ \frac{1}{2m} f(bw^-)(bw^-) - F(bw^-) \right] dx. \end{aligned}$$

Now, using (3.13) we get

$$\begin{aligned} &\int_{\Omega} \left[ \frac{1}{2m} f(aw^+)(aw^+) - F(aw^+) \right] dx \\ &+ \int_{\Omega} \left[ \frac{1}{2m} f(bw^-)(bw^-) - F(bw^-) \right] dx \\ &< \int_{\Omega} \left[ \frac{1}{2m} f(w^+)(w^+) - F(w^+) \right] dx \\ &+ \int_{\Omega} \left[ \frac{1}{2m} f(w^-)(w^-) - F(w^-) \right] dx \end{aligned}$$

which implies

$$h^w(a, b) < J(w^+ + w^-) = J(w) = h^w(1, 1)$$

and item (a) is proved.

Let us prove item (b). Consider the notations  $\Upsilon_1^w(t, s) = J'(tw^+ + sw^-)w^+$  and  $\Upsilon_2^w(t, s) = J'(tw^+ + sw^-)w^-$ . Thus,

$$\Upsilon_1^w(t, s) = M \left( \int_{\Omega} \Phi(t|\nabla w^+| + s|\nabla w^-|) dx \right) \int_{\Omega} \phi(t|\nabla w^+|) t |\nabla w^+|^2 dx - \int_{\Omega} f(tw^+) w^+ dx$$

and

$$\Upsilon_2^w(t, s) = M \left( \int_{\Omega} \Phi(t|\nabla w^+| + s|\nabla w^-|) dx \right) \int_{\Omega} \phi(s|\nabla w^-|) s |\nabla w^-|^2 dx - \int_{\Omega} f(sw^-) w^- dx.$$

Then

$$\begin{aligned} & \frac{\partial \Upsilon_1^w}{\partial t}(t, s) \\ &= M' \left( \int_{\Omega} \Phi(t|\nabla w^+| + s|\nabla w^-|) dx \right) \int_{\Omega} \Psi(t|\nabla w^+| + s|\nabla w^-|) |\nabla w^+| dx \int_{\Omega} \Psi(t|\nabla w^+|) |\nabla w^+| dx \\ & - \int_{\Omega} f'(tw^+) (w^+)^2 dx + M \left( \int_{\Omega} \Phi(t|\nabla w^+| + s|\nabla w^-|) dx \right) \int_{\Omega} \Psi'(t|\nabla w^+|) |\nabla w^+|^2 dx \end{aligned}$$

implies

$$\begin{aligned} & \frac{\partial \Upsilon_1^w}{\partial t}(1, 1) \\ &= M' \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \int_{\Omega} \Psi(|\nabla w|) |\nabla w^+| dx \int_{\Omega} \Psi(|\nabla w^+|) |\nabla w^+| dx \\ & - \int_{\Omega} f'(w^+) (w^+)^2 dx + M \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \int_{\Omega} \Psi'(|\nabla w^+|) |\nabla w^+|^2 dx \\ &\leq mM' \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \int_{\Omega} \Phi(|\nabla w|) dx \int_{\Omega} \Psi(|\nabla w^+|) |\nabla w^+| dx \\ & - \int_{\Omega} f'(w^+) (w^+)^2 dx + M \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \int_{\Omega} \Psi'(|\nabla w^+|) |\nabla w^+|^2 dx. \end{aligned}$$

Using (3.15) in the last equality we obtain

$$\begin{aligned} & \frac{\partial \Upsilon_1^w}{\partial t}(1, 1) \leq mM \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \int_{\Omega} \Psi(|\nabla w^+|) dx \\ & - \int_{\Omega} f'(w^+) (w^+)^2 dx + (m-1)M \left( \int_{\Omega} \Phi(|\nabla w|) dx \right) \int_{\Omega} \Psi(|\nabla w^+|) dx \\ & = (2m-1) \int_{\Omega} f(w^+) w^+ dx - \int_{\Omega} f'(w^+) (w^+)^2 dx. \end{aligned}$$

From (3.16) we obtain

$$\frac{\partial \Upsilon_1^w}{\partial t}(1, 1) < 0. \quad (3.25)$$

Arguing of the same way we conclude

$$\frac{\partial \Upsilon_2^w}{\partial s}(1, 1) < 0. \quad (3.26)$$

Since  $\frac{\partial \Upsilon_1^w}{\partial s}(1, 1) = 0$  and  $\frac{\partial \Upsilon_2^w}{\partial t}(1, 1) = 0$  and considering (3.25) and (3.26) we conclude that  $\det(\Upsilon^w)'(1, 1) = \frac{\partial \Upsilon_1^w}{\partial t}(1, 1) \frac{\partial \Upsilon_2^w}{\partial s}(1, 1) > 0$  and the item (b) is proved.  $\blacksquare$

## 4 Proof of Theorem 1.1

In this section we will prove the existence of  $w \in \mathcal{M}$  in which the infimum of  $J$  is attained on  $\mathcal{M}$ . After, following some arguments used in [5] by Alves and Souto (see also [9]) and, in particular, applying a deformation lemma, we find that  $w$  is a critical point of  $J$  and then a least energy nodal solution of  $(P)$ . In order to complete the proof of Theorem 1.1, we conclude by showing that  $w$  has exactly two nodal domains.

First of all, by Lemma 3.1, there exists  $c_0 \in \mathbf{R}$  such that

$$0 < c_0 = \inf_{v \in \mathcal{M}} J(v).$$

Thus, there exists a minimizing sequence  $(w_n)$  in  $\mathcal{M}$  which is bounded from Lemma 3.1. again. Hence, by Sobolev Imbedding Theorem, without loss of generality, we can assume up to a subsequence that there exist  $w, w_1, w_2 \in W_0^{1,\Phi}(\Omega)$  such that

$$\begin{aligned} w_n \rightharpoonup w, \quad w_n^+ \rightharpoonup w_1, \quad w_n^- \rightharpoonup w_2 \quad & \text{in } W_0^{1,\Phi}(\Omega), \\ w_n \rightarrow w, \quad w_n^+ \rightarrow w_1, \quad w_n^- \rightarrow w_2 \quad & \text{in } L^q(\Omega), \quad q \in (m, l^*). \end{aligned}$$

Since the transformations  $w \rightarrow w^+$  and  $w \rightarrow w^-$  are continuous from  $L^q(\Omega)$  in  $L^q(\Omega)$  (see Lemma 2.3 in [15] with suitable adaptations), we have that  $w^+ = w_1 \geq 0$  and  $w^- = w_2 \leq 0$ . At this point, we can prove that  $w \in \mathcal{M}$ . Indeed, by  $w_n^+ \rightarrow w^+$  and  $w_n^- \rightarrow w^-$  in  $L^q(\Omega)$  it is, as  $n \rightarrow +\infty$

$$\int_{\Omega} |(w_n)^{\pm}|^q dx \rightarrow \int_{\Omega} |w^{\pm}|^q dx.$$

Then, by Lemma 3.2, we conclude that  $w^{\pm} \neq 0$  and consequently  $w = w^+ + w^-$  is sign-changing. By Lemma 3.3, there exist  $t, s > 0$  such that

$$\begin{aligned} J'(tw^+ + sw^-)w^+ &= 0, \\ J'(tw^+ + sw^-)w^- &= 0, \end{aligned} \tag{4.1}$$

then  $tw^+ + sw^- \in \mathcal{M}$ . Now, let us prove that  $t, s \leq 1$ . First let us observe that, since  $f$  has a quasicritical growth, using compactness Lemma of Strauss [12, Theorem A.I, p.338], we obtain

$$\int_{\Omega} f((w_n)^{\pm})(w_n)^{\pm} dx \rightarrow \int_{\Omega} f(w^{\pm})w^{\pm} dx$$

and

$$\int_{\Omega} F((w_n)^{\pm}) dx \rightarrow \int_{\Omega} F(w^{\pm}) dx.$$

Thus, since  $J'(w_n)w_n^{\pm} = 0$ , by  $(M_1)$  we have

$$J'(w^+)w^+ \leq 0 \quad \text{and} \quad J'(w^-)w^- \leq 0. \tag{4.2}$$

Consequently, combining (4.1) and (4.2) and arguing as in the proof of Lemma 3.5 item (a), we obtain  $0 < t, s \leq 1$ .

In the next step we show that  $J(tw^+ + sw^-) = c_0$  and  $t = s = 1$  or better  $J(w) = c_0$ . Indeed, since  $t, s \leq 1$  and  $w_n \rightharpoonup w$  as  $n \rightarrow +\infty$ , exploiting the arguments used in the proof of Lemma 3.5 item (a) and the weak lower semicontinuity of both  $J$  and  $K$  with  $K(u) = J'(u)u$  on  $W_0^{1,\Phi}(\Omega)$  described above we get

$$\begin{aligned} c_0 \leq J(tw^+ + sw^-) &= J(tw^+ + sw^-) - \frac{1}{2m} J'(tw^+ + sw^-)(tw^+ + sw^-) \\ &\leq J(w^+ + w^-) - \frac{1}{2m} J'(w^+ + w^-)(w^+ + w^-) \\ &\leq \liminf_{n \rightarrow +\infty} \left[ J(w_n^+ + w_n^-) - \frac{1}{2m} J'(w_n^+ + w_n^-)(w_n^+ + w_n^-) \right] \\ &= \lim_{n \rightarrow +\infty} J(w_n) = c_0. \end{aligned}$$

At this point, by using a quantitative deformation lemma and adapting the arguments used in [9] with slight technical changes, we point out that  $w$  is a critical point of  $J$ , i.e.  $J'(w) = 0$ . If we reason by contradiction, we find that there exist a positive constant  $\alpha > 0$  and  $v_0 \in W_0^{1,\Phi}(\Omega)$ ,  $\|v_0\| = 1$  such that

$$J'(w)v_0 = 2\alpha > 0.$$

By the continuity of  $J'$ , we can choose a radius  $r > 0$  so that

$$J'(v), v_0 = \alpha > 0, \text{ for every } v \in B_r(w) \subset W_0^{1,\Phi}(\Omega) \text{ with } v^\pm \neq 0.$$

Let us fix  $D = (\xi, \chi) \times (\xi, \chi) \subset \mathbf{R}^2$  with  $0 < \xi < 1 < \chi$  such that

- (i)  $(1, 1) \in D$  and  $\Upsilon^w(t, s) = (0, 0)$  in  $\overline{D}$  if and only if  $(t, s) = (1, 1)$ ;
- (ii)  $c_0 \notin h^w(\partial D)$ ;
- (iii)  $\{tw_0^+ + sw_0^- : (t, s) \in \overline{D}\} \subset B_r(w)$ ,

where  $h^w$  and  $\Upsilon^w$  are defined as in Section 3 and satisfy Lemma 3.5. At this point, we can choose a smaller radius  $r' > 0$  such that

$$\mathcal{B} = \overline{B_{r'}(w)} \subset B_r(w) \text{ and } \mathcal{B} \cap \{tw^+ + sw^- : (t, s) \in \partial D\} = \emptyset. \quad (4.3)$$

Now define a continuous mapping  $\rho : W_0^{1,\Phi}(\Omega) \rightarrow [0, +\infty)$  such that

$$\rho(u) := \text{dist}(u, \mathcal{B}^c), \text{ for all } u \in W_0^{1,\Phi}(\Omega),$$

then a bounded Lipschitz vector field  $V : W_0^{1,\Phi}(\Omega) \rightarrow W_0^{1,\Phi}(\Omega)$  given by

$$V(u) = -\rho(u)v_0$$

and, for every  $u \in W_0^{1,\Phi}(\Omega)$ , denoting by  $\eta(\tau) = \eta(\tau, u)$  we consider the following Cauchy problem

$$\begin{cases} \eta'(\tau) = V(\eta(\tau)), & \text{for all } \tau > 0, \\ \eta(0) = u. \end{cases}$$

Now, we observe that there exist a continuous deformation  $\eta(\tau, u)$  and  $\tau_0 > 0$  such that for all  $\tau \in [0, \tau_0]$  the following properties hold:

- (a)  $\eta(\tau, u) = u$  for all  $u \notin \mathcal{B}$ ;
- (b)  $\tau \rightarrow J(\eta(\tau, u))$  is decreasing for all  $\eta(\tau, u) \in \mathcal{B}$ ;
- (c)  $J(\eta(\tau, w_0)) \leq J(w) - \frac{r'\alpha}{2}\tau$ .

Item (a) follows immediately by the definition of  $\rho$ . Indeed,  $u \notin \mathcal{B}$  implies  $\rho(u) = 0$  and the unique solution satisfying the above Cauchy problem is constant with constant value  $u$ . As concerns as item (b), let us first observe that, since  $\eta(\tau) \in \mathcal{B} \subset B_r(w)$ ,  $J'(\eta(\tau))v_0 = \alpha > 0$  and, by definition of  $\rho$ , it is  $\rho(\eta(\tau)) > 0$ . Now, differentiating  $J$  with respect to  $\tau$ , for all  $\eta(\tau) \in \mathcal{B}$ , we have that

$$\frac{d}{d\tau} (J(\eta(\tau))) = J'(\eta(\tau))\eta'(\tau) = -\rho(\eta(\tau))J'(\eta(\tau))v_0 = -\rho(\eta(\tau))\alpha < 0$$

thus concluding that  $J(\eta(\tau, u))$  is decreasing with respect to  $\tau$ .

In order to prove item (c), being  $\tau_0 > 0$  such that  $\eta(\tau, u) \in \mathcal{B}$  for every  $0 \leq \tau \leq \tau_0$ , we can assume without loss of generality

$$\|\eta(\tau, w) - w\| \leq \frac{r'}{2} \iff \eta(\tau, w) \in \overline{B_{\frac{r'}{2}}(w)}, \text{ for every } 0 \leq \tau \leq \tau_0.$$

Thus, since  $\rho(\eta(\tau, w)) = \text{dist}(\eta(\tau, w), \mathcal{B}^c) \geq \frac{r'}{2}$  it follows that

$$\frac{d}{d\tau} (J(\eta(\tau, w))) = -\rho(\eta(\tau, w))\alpha \leq -\frac{r'\alpha}{2}$$

and, integrating in  $[0, \tau_0]$  we finally get

$$J(\eta(\tau, w_0)) - J(w) \leq -\frac{r'\alpha}{2}\tau.$$

At this point, let us consider a suitable deformed path  $\bar{\eta}_0 : \bar{D} \rightarrow X$  defined by

$$\bar{\eta}_{\tau_0}(t, s) := \eta(\tau_0, tw^+ + sw^-), \text{ for all } (t, s) \in \bar{D}$$

so that

$$\max_{(t,s) \in \bar{D}} J(\bar{\eta}_{\tau_0}(t, s)) < c_0.$$

Indeed, by (b) and the fact that  $\eta$  satisfies the initial condition  $\eta(0, u) = u$ , for all  $(t, s) \in \bar{D} - \{(1, 1)\}$  it is

$$\begin{aligned} J(\bar{\eta}_{\tau_0}(t, s)) &= J(\eta(\tau_0, tw^+ + sw^-)) \leq J(\eta(0, tw^+ + sw^-)) \\ &= J(tw^+ + sw^-) = h^w(t, s) < c_0, \end{aligned}$$

and, for  $(t, s) = (1, 1)$ , by (c) we get

$$\begin{aligned} J(\bar{\eta}_{\tau_0}(1, 1)) &= J(\eta(\tau_0, w^+ + w^-)) = J(\eta(\tau_0, w)) \\ &\leq J(w) - \frac{r'\alpha}{2}\tau_0 < J(w) < c_0. \end{aligned}$$

Then,  $\bar{\eta}_{\tau_0}(\bar{D}) \cap \mathcal{M} \neq \emptyset$ , i.e.

$$\bar{\eta}_{\tau_0}(t, s) \notin \mathcal{M}, \text{ for all } (t, s) \in \bar{D}. \quad (4.4)$$

On the other side, defined  $\Lambda_{\tau_0} : \bar{D} \rightarrow \mathbf{R}^2$  such that

$$\Lambda_{\tau_0} := \left( \frac{J'(\bar{\eta}_{\tau_0}(t, s))(\bar{\eta}_{\tau_0}(t, s))^+}{t}, \frac{J'(\bar{\eta}_{\tau_0}(t, s))(\bar{\eta}_{\tau_0}(t, s))^-}{s} \right),$$

we observe that, for all  $(t, s) \in \partial D$ , by (4.3) and (a) for  $\tau = \tau_0$ , it is

$$\Lambda_{\tau_0}(t, s) = \left( J'(tw^+ + sw^-)w^+, J'(tw^+ + sw^-), w^- \right) = \Phi^w(t, s).$$

Then, since by Brouwer's topological degree

$$\deg(\Lambda_{\tau_0}, D, (0, 0)) = \deg(\Phi^w, D, (0, 0)) = \text{sgn}(\det(\Phi^w)'(1, 1)) = 1,$$

we get that  $\Lambda_{\tau_0}$  has a zero  $(\bar{t}, \bar{s}) \in D$  namely

$$\Lambda_{\tau_0}(\bar{t}, \bar{s}) = (0, 0) \iff J'(\bar{\eta}_{\tau_0}(\bar{t}, \bar{s}))(\bar{\eta}_{\tau_0}(\bar{t}, \bar{s}))^\pm = 0.$$

Consequently there exists  $(\bar{t}, \bar{s}) \in D$  such that  $\bar{\eta}_{\tau_0}(\bar{t}, \bar{s}) \in \mathcal{M}$  and we have a contradiction with (4.4). We conclude that  $w$  is a critical point of  $J$ .

Finally, we prove that  $w$  has exactly two nodal domains or equivalently it changes sign exactly once. Let us observe that assumptions  $(M_1)$ ,  $(f_1)$  and  $(f_2)$  ensure that  $w$  is continuous and then  $\tilde{\Omega} = \{x \in \Omega : w(x) \neq 0\}$  is open. Suppose by contradiction that  $\tilde{\Omega}$  has more than two components or  $w$  has more than two nodal domains and, since  $w$  changes sign, without loss of generality, we can assume

$$w = w_1 + w_2 + w_3, \text{ where } w_1 \geq 0, w_2 \leq 0, w_3 \neq 0,$$

and

$$\text{supp}(w_i) \cap \text{supp}(w_j) = \emptyset, \text{ for } i \neq j, i, j = 1, 2, 3.$$

So the disjointness of the supports combined with  $J'(w) = 0$  implies

$$\langle J'(w_1 + w_2), w_1 \rangle = 0 = \langle J'(w_1 + w_2), w_2 \rangle.$$

Since  $0 \neq w_1 = (w_1 + w_2)^+$  and  $0 \neq w_2 = (w_1 + w_2)^-$ , by previous arguments, there exist  $t, s \in (0, 1]$  such that  $t(w_1 + w_2)^+ + s(w_1 + w_2)^- \in \mathcal{M}$  namely  $tw_1 + sw_2 \in \mathcal{M}$  and then  $J(tw_1 + sw_2) \geq c_0$ .

On the other side,  $0 \neq w_3 \in \mathcal{N}$ , Lemma 2.1 (i) and the arguments used in the proof of Lemma 3.5 imply that

$$J(tw_1 + sw_2) \leq J(w_1 + w_2) < J(w_1 + w_2) + J(w_3) = J(w) = c_0$$

then a contradiction and we conclude that  $w_3 = 0$ . Thus, the proof of Theorem 1.1 is complete.  $\blacksquare$

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